AN INTEGRAL FORMULA

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The following results generalize in several directions a recent formula of Richard Kraft [3, Lemma 1] used in a problem of geometrical optics.

Theorem 1. *Let M be a compact n-manifold imbedded in a Euclidean $(n + 1)$-space E^{n+1}, where $n > 1$. For each $x \in M$, let $e = e(x)$ be the outward unit normal, $r = |x|$, and $p = p(x) = x \cdot e$, the support function. Also let σ denote the element of n-volume. Then*

$$\frac{1}{V_n} \int_M \frac{px}{r^{n+2}} \sigma = \begin{cases} 0 & \text{if } 0 \notin M , \\ -e(0) & \text{if } 0 \in M , \end{cases}$$

where $V_n = \pi^{n/2}/\Gamma(\tfrac{1}{2}n + 1)$ is the volume of the unit n-ball.

Our proof will be based on two formal lemmas. We shall denote by $[v_1, \cdots, v_n]$ the cross (vector) product of n vectors in E^{n+1}, assumed oriented. As usual, we extend this alternating multilinear function to vectors with differential form coefficients by

$$[\alpha_1 v_1, \cdots, \alpha_n v_n] = (\alpha_1 \wedge \cdots \wedge \alpha_n)[v_1, \cdots, v_n] .$$

We refer to Flanders [1, pp. 43, 149] and [2] for this formalism.

Lemma 1. *On M we have*

$$n(x \cdot dx) \wedge [x, dx, \cdots, dx] = r^2[dx, \cdots, dx] - n! \, px\sigma .$$

Proof. We shall give more detail than is really necessary, because the probability of an error in sign is high in calculations of this type.

Let $e_1, \cdots, e_n$ be a moving orthonormal frame on M, so

$$x = p_i e_i + pe , \qquad dx = \sigma_i e_i ,$$

where the σ_i are one-forms, and repeated indices are summed. Note that $\sigma_1 \wedge \cdots \wedge \sigma_n = \sigma$ is the volume element on M. We take the e_i so that $e_1, \cdots, e_n, e$ is a right-handed frame for E^{n+1}. Then $[e_1, \cdots, e_n] = e$. We also note for future reference that

$$[e, e_1, \cdots, \hat{e}_i, \cdots, e_n] = (-1)^i e_i ,$$

because it requires $(n - i) + n$ transpositions to pass from $e, e_1, \cdots, \hat{e}_i, \cdots,$ e_n, e_i to $e_1, \cdots, e_n, e$. (The circumflex denotes omission.)

We have

$$(x \cdot dx) \wedge [x, dx, \cdots, dx]$$
$$= (p_i \sigma_i) \wedge \{[p_i e_i, \sigma_j e_j, \cdots, \sigma_k e_k] + p[e, \sigma_j e_j, \cdots, \sigma_k e_k]\} .$$

Now

$$[p_i e_i, \sigma_j e_j, \cdots, \sigma_k e_k]$$
$$= p_i(\sigma_j \wedge \cdots \wedge \sigma_k)[e_i, e_j, \cdots, e_k]$$
$$= (n - 1)! \sum_i p_i(\sigma_1 \wedge \cdots \wedge \hat{\sigma}_i \wedge \cdots \wedge \sigma_n)[e_i, e_1, \cdots \hat{e}_i, \cdots, e_n]$$
$$= (n - 1)! \left(\sum_i (-1)^{i-1} p_i \sigma_1 \wedge \cdots \wedge \hat{\sigma}_i \wedge \cdots \wedge \sigma_n \right) e ,$$

hence

$$(p_i \sigma_i) \wedge [p_i e_i, \sigma_j e_j, \cdots, \sigma_k e_k] = (n - 1)! (\sum p_i^2) \sigma e .$$

Next,

$$[e, \sigma_j e_j, \cdots, \sigma_k e_k]$$
$$= (\sigma_j \wedge \cdots \wedge \sigma_k)[e, e_j, \cdots, e_k]$$
$$= (n - 1)! \sum_i (\sigma_1 \wedge \cdots \wedge \hat{\sigma}_i \wedge \cdots \wedge \sigma_n)[e, e_1, \cdots, \hat{e}_i, \cdots, e_n]$$
$$= (n - 1)! \sum_i (-1)^i (\sigma_1 \wedge \cdots \wedge \hat{\sigma}_i \wedge \cdots \wedge \sigma_n) e_i ,$$

hence

$$(p_i \sigma_i) \wedge [e, \sigma_j e_j, \cdots, \sigma_k e_k] = -(n - 1)! \, \sigma(p_i e_i) .$$

Consequently

$$(x \cdot dx) \wedge [x, dx, \cdots, dx] = (n - 1)! \, [(\sum p_i^2) e - p(p_i e_i)] \sigma$$
$$= (n - 1)! \, (r^2 e - px) \sigma .$$

Since $[dx, \cdots, dx] = n! \, \sigma e$, the lemma follows.

Lemma 2.   *On M we have*

$$n! \, r^{-(n+2)} px \sigma = d(r^{-n}[x, dx, \cdots, dx]) .$$

Proof.   Applying d and using Lemma 1 we obtain

$$d(r^{-n}[x, dx, \cdots, dx])$$
$$= -nr^{-(n+2)}(x \cdot dx) \wedge [x, dx, \cdots, dx] + r^{-n}[dx, \cdots, dx]$$
$$= -r^{-n}[dx, \cdots, dx] + n! \, r^{-(n+2)}px\sigma + r^{-n}[dx, \cdots, dx]$$
$$= n! \, r^{-(n+2)}px\sigma \; .$$

Proof of Theorem 1. If $0 \notin M$, then the integrand is exact on M by Lemma 2, so the integral is zero. If $0 \in M$, the integrand is singular at 0. We choose ε so small that $\{r = \varepsilon\} \cap M$ is an $(n-1)$-sphere and set $M_\varepsilon = M \backslash \{r < \varepsilon\}$. By the lemma and two applications for Stokes's theorem,

$$\int_{M_\varepsilon} \frac{px}{r^{n+2}}\sigma = \frac{1}{n!} \int_{M_\varepsilon} d\left(\frac{1}{r^n}[x, dx, \cdots, dx]\right)$$
$$= \frac{1}{n!} \int_{\partial M_\varepsilon} \frac{1}{r^n}[x, dx, \cdots, dx] = -\frac{1}{n!} \int_{r=\varepsilon} \frac{1}{r^n}[x, dx, \cdots, dx]$$
$$= \frac{-1}{n! \, \varepsilon^n} \int_{r=\varepsilon} [x, dx, \cdots, dx] = \frac{-1}{n! \, \varepsilon^n} \int_{r \le \varepsilon} [dx, dx, \cdots, dx]$$
$$= \frac{-1}{n! \, \varepsilon^n} \int_{r \le \varepsilon} n! \, e\sigma \approx \frac{-1}{\varepsilon^n}e(0) \int_{r \le \varepsilon} \sigma \approx \frac{-1}{\varepsilon^n}(\varepsilon^n V_n)e(0) \to - V_n e(0) \; .$$

It is clear that the convergence is absolute as $\varepsilon \to 0$ so that $M_\varepsilon \to M$. Therefore any other family M_ε' converging to M would yield the same value for the singular integral.

Corollary. *If $x_0 \in M$, then*

$$\frac{1}{V_n} \int_M \frac{(x - x_0) \cdot e}{|x - x_0|^{n+2}} x d\sigma = -e(x_0) \; .$$

These results can be extended without difficulty to immersed rather than imbedded orientable hypersurfaces. The case of a closed curve is special.

Theorem 2. *Let C be a simple closed smooth counter-clockwise curve in E^2 with Frenet frame t, n at x. Set $p = -x \cdot n$, $r = |x|$, and J the $90°$ rotation. Then*

$$\frac{1}{2} \int_C \frac{p}{r^3} J(x) ds = \begin{cases} 0 & \text{if } 0 \notin C \; , \\ -t(0) & \text{if } 0 \in C \; . \end{cases}$$

Proof. Write $x = at - pn$. Then $dx = t ds$, $r^2 = a^2 + p^2$, $r dr = x \cdot dx = a ds$, and

$$d(r^{-1}x) = -ar^{-3}ds x + r^{-1}ds t = r^{-3}[-a(at - pn) + (a^2 + p^2)t]ds$$
$$= pr^{-3}(pt + an)ds = pr^{-3}J(x)ds \; .$$

The theorem follows easily.

References

[1]   H. Flanders, *Differential forms with applications to the physical sciences*, Academic Press, New York, 1963.

[2]   ——, *The Steiner point of a closed hypersurface*, Mathematika **13** (1966) 181–186.

[3]   R. Kraft, *Uniqueness and existence for the integral equation of interreflections*, SIAM J. Math. Anal., to appear.

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